Transonic Lifting Line Theory—Numerical Procedure for Shock-Free Flows

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Nomenclature

A	= similarity parameter, α/δ
$A\!\!R$	= aspect ratio
b	= half-span
B	= perturbation parameter, $B=\delta^{1/3}b$
$c(z^*)$	= chord distribution
$c_{i,j}$	= coefficients defined by Eq. (11)
c_1,c_2	= coefficients defined by Eq. (12b)
C_p	= pressure coefficient
C_L	= lift coefficient
F,G	= functions defining wing cross section
h	= mesh spacing
K	= transonic similarity parameter
L	= lift
M	= Mach number
r	= polar coordinate, $r^2=x^2+K\tilde{y}^2$
U	= velocity
x	= Cartesian coordinate
$\tilde{y},\tilde{z},z^*$	= scaled coordinates
α	= angle of attack
β	= compressibility factor, $\beta=\sqrt{1-M^2}$
γ	= ratio of specific heats
Γ	= circulation
δ	= airfoil thickness ratio
θ	= polar coordinate, $\theta=\tan^{-1}(\sqrt{K}\tilde{y}/x)$
ξ	= dummy variable
ρ	= fluid density
ϕ	= disturbance potential
ϕ^*	= transformed first-order disturbance potential
Φ	= exact potential

Subscripts

i,j	= mesh point indices in x and $\tilde{y}$ directions, respectively
L	= wing leading edge
T	= wing trailing edge
u,ℓ	= denotes upper and lower airfoil surfaces
$x,\tilde{y}$	= denotes derivative with respect to $x,\tilde{y}$
$0,1$	= denotes zero- and first-order terms
∞	= freestream

I. Introduction

RECENTLY, numerical methods have been developed for solving the inviscid flow equations for slender airfoils in transonic flows. Present extensions of the two-dimensional numerical methods include calculation of transonic flows about three-dimensional wings. In the following, an algorithm for calculation of a transonic lifting line theory is developed and numerical results for an elliptic wing with a NACA-0012 cross section are presented. The calculated lift coefficients are compared with results—corrected for compressibility effects—obtained from incompressible lifting line theory.

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II. Definition of the Boundary Value Problem

The equations and linearized boundary conditions describing the transonic ($M_\infty \sim 1$) flow about an unswept finite-aspect-ratio wing at small angle of attack can be developed[1] from the three-dimensional transonic small disturbance equation

$$(K-(\gamma+1)\phi_x)\phi_{xx}+\phi_{\tilde{y}\tilde{y}}+\phi_{\tilde{z}\tilde{z}}=0 \tag{1}$$

with boundary conditions and Kutta condition

$$\phi_{\tilde{y}}(x,0,\tilde{z})=\frac{\partial}{\partial x}F_{u,\ell}(x,\tilde{z})-A,\ \ x_L(\tilde{z})\le x\le x_T(\tilde{z}) \tag{2a}$$

$$\phi_x(x,\tilde{y},\tilde{z},),\quad \phi_{\tilde{y}}(x,\tilde{y},\tilde{z})\to 0,\quad r\to\infty \tag{2b}$$

$$[\phi_x(x,0,\tilde{z})]=0,\quad x=x_T(\tilde{z}) \tag{2c}$$

Using the method of matched asymptotic expansions, the preceding three-dimensional transonic disturbance equation is reduced to the following two-dimensional boundary value problems (in which span effects appear only parametrically):

1)

$$(K-(\gamma+1)\phi_{0x})\phi_{0xx}+\phi_{0\tilde{y}\tilde{y}}=0 \tag{3}$$

$$\phi_{0\tilde{y}}(x,0^\pm,z^*)=\frac{\partial}{\partial x}F_{u,\ell}(x,z^*)-A,$$

$$|z^*|\le 1,\quad x_L(z^*)\le x\le x_T(z^*) \tag{4a}$$

$$\phi_0(x,\tilde{y},\tilde{z}^*)\to -\frac{\Gamma_0(z^*)}{2\pi}\theta+\frac{(\gamma+1)\Gamma_0^2(z^*)}{16\pi^2K}\frac{\ell n r}{r}\cos\theta,\ \ r\to\infty \tag{4b}$$

$$[\phi_{0x}(x,0,z^*)]=0,\ \ x=x_T(z^*),\ \ |z^*|\le 1 \tag{4c}$$

2)

$$(K-(\gamma+1)\phi_{0x})\phi^*_{xx}-(\gamma+1)\phi_{0xx}\phi^*_x+\phi^*_{\tilde{y}\tilde{y}}=0 \tag{5a}$$

$$\phi^*(x,\tilde{y},z^*)=\phi_1(x,\tilde{y},z^*)+\frac{\tilde{y}}{2\pi}\oint_{-1}^{1}\frac{\Gamma_0'(\xi)\mathrm{d}\xi}{z^*-\xi} \tag{5b}$$

$$\phi^*_{\tilde{y}}(x,0,z^*)=\frac{1}{2\pi}\oint_{-1}^{1}\frac{\Gamma_0'(\xi)\mathrm{d}\xi}{z^*-\xi},\ \ x_L(z^*)\le x\le x_T(z^*),$$

$$|z^*|\le 1 \tag{6a}$$

$$\phi^*(x,\tilde{y},z^*)\to -\frac{\Gamma_1(z^*)}{2\pi}\theta,\quad r\to\infty \tag{6b}$$

$$[\phi^*_x(x,0,z^*)]=0,\quad x\ge x_T(z^*),\quad |z^*|\le 1 \tag{6c}$$

ϕ_0 and ϕ_1 are the disturbance potentials from the inner expansion

$$\phi(x,\tilde{y},z^*;K,A,B)=\phi_0(x,\tilde{y},z^*;K,A)$$

$$+(1/B)\phi_1(x,\tilde{y},z^*;K,A)+\ldots \tag{7}$$

Equation (3) with the tangent flow condition [Eq. (4a)] on the wing, far-field behavior [Eq. (4b)], and Kutta condition, [Eq. (4c)] represents a two-dimensional section flow. Numerical methods for its solution are well documented.[2,3] The particular algorithm used in the present study is described in Ref. 4. The solution of the linear homogeneous Eq. (5a) subject to the downwash [Eq. (6a)] on the wing, far-field condition [Eq. (6b)], and Kutta condition [Eq. (6c)] represents the correction for finite span to two-dimensional wing (transonic) theory.

The solution of Eq. (1) with boundary conditions [Eq. (2)] therefore reduces to construction of a three-dimensional potential field from a series of solutions to two-dimensional problems in which span effects appear parametrically. At each cross section, z_k^*, $\phi_0(x,\tilde{y},z^*)$, and $\Gamma_0(z^*)$ are to be determined from Eqs. (3) and (4), and then Eqs. (5) and (6) are to be solved for the first-order inner disturbance potential and circulation $\phi_1(x,\tilde{y},z^*)$ and $\Gamma_1(z^*)$. For the special case of similar profiles, i.e.,

$$F_{u,\ell}(x,z^*) - Ax = c(z^*)G_{u,\ell}[x/c(z^*)] \tag{8}$$

where $c(z^*)$ defines the chord, the three-dimensional field may be extrapolated by a scaling[1] of ϕ_0 and ϕ_1 (which are calculated at one cross section, e.g., center span).

For the following definitions of $\tilde{y}$, z^*, and K

$$\tilde{y} = \delta^{1/3}M_\infty^{1/2}y, \quad z^* = z/b, \quad K = (1-M_\infty^2)/M_\infty\delta^{2/3} \tag{9}$$

the pressure coefficient and lift are

$$C_p = -\frac{2\delta^{2/3}}{M_\infty^{3/4}}\left(\phi_{0x} + \frac{1}{B}\phi_{1x}\right) \tag{10a}$$

$$L = \rho U_\infty^2 b\frac{\delta^{2/3}}{M_\infty^{3/4}}\int_{-1}^{1}[\Gamma_0(z^*) + \frac{1}{B}\Gamma_1(z^*)]\mathrm{d}z^* \tag{10b}$$

III. Numerical Method

A converged numerical solution to the posed boundary value problem is obtained by establishing the potential fields ϕ_0 and ϕ_1 in the infinite section planes which satisfy the boundary conditions in Eqs. (4a,b) and (6a,b), respectively. Unique solutions are prescribed by the Kutta condition in Eqs. (4c) and (6c). Using the far-field representations [Eqs. (4b) and (6b)], it was sufficient to solve for ϕ_0 and ϕ_1 in the plane $|x| \le 3$, $|\tilde{y}| \le 6$. The algorithm for solution of ϕ_0 and ϕ_1 is summarized as follows:

1) The two-dimensional transonic small disturbance equation (3) is solved in the plane $|x| \le 3$, $|\tilde{y}| \le 6$, z^* = constant. The potential $\phi_0(x,\tilde{y},z^*)$ is specified [Eq. (4b)] on the far-field boundary, and on the slit ($\tilde{y}=0$, $x_L(z^*) \le x \le x_T(z^*)$, $|z^*| \le 1$), $\phi_{0\tilde{y}}(x,0^\pm,z^*)$ are given by Eq. (4a). Across the cut $\tilde{y}=0$, $x > x_T(z^*)$, $|z^*| \le 1$, the potential jumps in value by the circulation $\Gamma_0(z^*)$. The Kutta condition [Eq. (4c)] guarantees a unique solution.

2) Having computed $\phi_0(x,\tilde{y},z^*)$ and $\Gamma_0(z^*)$ in the three-dimensional field by solving a series (at different z_k^*) of two-dimensional problems, the downwash velocity $\phi_{\tilde{y}}^*$ [Eq. (6a)] on the wing as well as the coefficients of ϕ_{xx}^* and ϕ_x^* are calculated.

3) The first-order potential equation is then solved in the plane $|x| \le 3$, $|\tilde{y}| \le 6$, z^* = constant. $\phi_{\tilde{y}}^*(x,0,z^*)$ is specified on the slit $x_L(z^*) \le x \le x_T(z^*)$, $|z^*| \le 1$ by Eq. (6a) and $\phi^*(x,\tilde{y},z^*)$ by Eq. (6b) in the far field. ϕ^* is also multivalued and across the cut $\tilde{y}=0$, $x > x_T(z^*)$, $|z^*| \le 1$ jumps in value $\Gamma_1(z^*)$. A unique circulation and, hence, solution is ensured by the Kutta condition [Eq. (6c)]. The three-dimensional solution $\Phi = U_\infty\{x + \delta^{2/3}[\phi_0 + (1/B)\phi_1] + \ldots\}$ is constructed by addition of ϕ_0 and ϕ_1 from k section calculations z_k^* = constant.

Both Eqs. (3) and (5a) are type-dependent and require selective finite-difference procedures. Equation (3) is the more difficult to solve as it is nonlinear and the type of the equation and hence finite-difference relation used is dependent on the local and current solution. The type of Eq. (5a) depends on the coefficient of ϕ_{xx}^* which is a priori known at each point in the field independent of the local value of ϕ^*. The specific differencing procedure used depends on the relation between

$$c_{i,j} = K - (\gamma+1)(\phi_{0_{i+1,j}} - \phi_{0_{i-1,j}})/2h \tag{11}$$

and $c_{i-1,j}$, i.e., $c_{i,j}$ evaluated at the adjacent upstream point. Standard difference operators[3] are used except for $c_{i,j} > 0$, $c_{i-1,j} < 0$ (isentropic deceleration to subsonic flow) where a smoothing operator

$$\phi_{\tilde{y}\tilde{y}}^* = \frac{-c_1(\phi_{i+1,j}^* - \phi_{i,j}^*) + c_2(\phi_{i-1,j}^* + \phi_{i-2,j}^*)}{4h^2} \tag{12a}$$

$$c_1 = \tfrac{1}{2}(c_{i,j} + c_{i+1,j}), \quad c_2 = \tfrac{1}{2}(c_{i-1,j} + c_{i-2,j}) \tag{12b}$$

similar to Murman's shock point operator[5] is used.

A converged solution for ϕ^* is obtained by a successive line relaxation procedure in which elliptic points are over-relaxed and hyperbolic and parabolic points are under-relaxed. A nonuniform mesh was used in the actual computations in order to have a higher density of mesh points near the wing. Values of $\phi_{i,j,k}^*$ interior to the boundaries are determined serially on lines of constant x by solution of a diagonally dominant tridiagonal matrix. At the wing trailing edge, the circulation $\Gamma_1(z^*)$ is redefined as $[\phi^*]$ and the far-field boundary condition [Eq. (6b)] reset. A converged solution is obtained when the maximum changes in ϕ^* and Γ_1 occurring in the field are less than some a priori defined error limit, e.g., $\epsilon = O(10^{-4})$. Since Eq. (5a) is linear, an infinity of solutions exist which satisfy Eqs. (6a) and (6b). The unique solution is that which satisfied Eq. (6c) [to $O(10^{-5})$], i.e., zero pressure jump across the trailing vortex sheet.

IV. Results

The three-dimensional flowfield was computed for a geometrically and aerodynamically untwisted lifting wing with an elliptic (spanwise) distribution of chord and a NACA-0012 cross section. The freestream Mach number and angle of attack were 0.63 and 2 deg, respectively.

Since all section profiles are similar, the three-dimensional field can be scaled from the values of ϕ_0 and ϕ^* computed at center span ($z^*=0$). For a nondimensional chord length at $z^*=0$ of 2, the scaling factor $\tfrac{1}{2}c(z^*)$ is $\sqrt{1-z^{*2}}$. The zero-order disturbance potential and circulation are therefore

$$\phi_0(x,\tilde{y},z^*) = \sqrt{1-z^{*2}}\,\phi_0(x,\tilde{y},0), \quad |z^*| \le 1$$

$$\Gamma_0(z^*) = \sqrt{1-z^{*2}}\,\Gamma_0(0), \quad |z^*| \le 1$$

The downwash velocity as calculated from the principle value integral [Eq. (6a)], the first-order disturbance potential, [Eq. (5b)] and first-order circulation are

$$\phi_{\tilde{y}}^* = \frac{1}{2\pi}\oint_{-1}^{1}\frac{\Gamma_0'(\xi)\mathrm{d}\xi}{z^*-\xi} = \frac{\Gamma_0(0)}{2}$$

$$\phi_1(x,\tilde{y},z^*) = \sqrt{1-z^{*2}}\left(\phi^*(x,\tilde{y},0) - \tilde{y}\frac{\Gamma_0(0)}{2}\right)$$

$$\Gamma_1(z^*) = \sqrt{1-z^{*2}}\,\Gamma_1(0)$$

$\phi_0(x,\tilde{y},0)$ and $\Gamma_0(0)$ were computed using the algorithm described in Ref. 4. The coefficient terms $K-(\gamma+1)\phi_{0x}$ and $(\gamma+1)\phi_{0xx}$ in Eq. (5a) were computed using central differences and stored on disk for use in solving Eq. (5a) and calculating the final total pressure distribution. A converged unique solution[6] for $\phi^*(x,\tilde{y},0)$ was obtained (initial field $\phi^*=0$ everywhere) after 287 iterations. The calculations were performed using double precision arithmetic, the program size was 270k bytes and required 80 seconds of execution time (IBM 360/91).

The total pressure distribution

$$C_p = -\frac{2\delta^{2/3}}{M_\infty^{3/4}}\left(\phi_{0x} + \frac{1}{\delta^{1/3}}\frac{4}{\pi}\frac{1}{\mathcal{R}}\phi_{1x}\right) \tag{13}$$

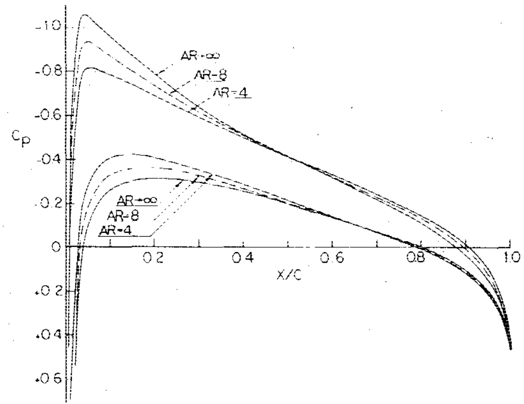

Fig. 1 Pressure distribution: elliptic wing, NACA-0012 cross section, $M_\infty = 0.63$, $\alpha = 2$ deg.

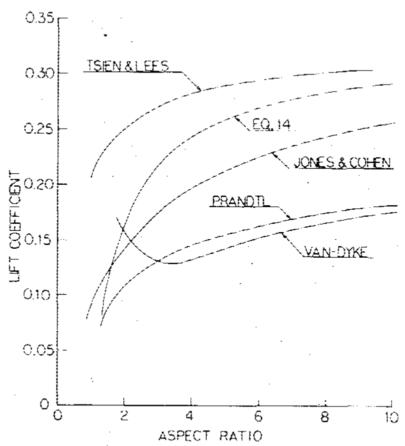

Fig. 2 Comparison of computed lift coefficients with incompressible lift coefficients corrected for compressibility effects.

for the elliptic NACA-0012 wing is shown in Fig. 1 for several different aspect ratio wings. The pressure distribution for the infinite wing showed excellent agreement with a solution[7] to the full inviscid equations. The lift coefficient as calculated from Eq. (10b) reduces to

$$C_L = \frac{\delta^{2/3}}{M_\infty^{3/4}}\left[\Gamma_0(0) + \frac{1}{\delta^{1/3}}\frac{4}{\pi}\frac{1}{A\!\!\!R}\Gamma_1(0)\right] \tag{14a}$$

$$C_L = 0.324 - 0.314/A\!\!\!R \tag{14b}$$

The first term represents the lift coefficient for a two-dimensional wing and the second term the correction due to finite aspect ratio. It should be noted that these results are derived from an asymptotic theory and are therefore formally valid as $A\!\!\!R \to \infty$. For an aspect ratio of 8 the lift of the finite wing is 13.7% less than the infinite aspect ratio wing.

While wind-tunnel test results are not available for comparison with these results, it is interesting to compare the computed lift coefficients to those calculated from incompressible lifting line theory corrected for compressibility effects. One method for correcting Prandtl's incompressible theory is based on a simple stretching of the wing chord by $1/\beta$. The corrected lift coefficient[8] is

$$C_L = 2\pi\alpha A\!\!\!R/(\beta A\!\!\!R + 2) \tag{15}$$

A different correction

$$C_L = 2\pi\alpha A\!\!\!R/[\beta A\!\!\!R + 2(1-\beta)] \tag{16}$$

in which the effective angle of attack is also modified has been proposed by Tsien and Lees.[9] Both these corrections are based on a stretching of the chord by $1/\beta$ and ignore thickness effects. The lift coefficients at infinite $A\!\!\!R$ disagree with the two-dimensional calculation. Multiplication of Eqs. (15) and (16) by $0.324/2\pi\alpha/\beta$ provides the correct asymptotic limit. The revised corrections and the computed result [Eq. (14)] are shown in Fig. 2. For comparison, results from Prandtl's and Van Dyke's[10] incompressible theories are shown.

V. Summary

The algorithm described is appropriate for computation of shock-free transonic flows about unswept wings. In addition to the subcritical flow calculated, supercritical flows can also be computed providing they are shock-free flows. The airfoil sections designed by Garabedian and Korn[11] and at NLR[12] fit this class. At present, the computer code is being modified to calculate flows with embedded shock waves.

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Pressure Pulsations on a Flat Plate Normal to an Underexpanded Supersonic Jet

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Introduction

THIS Note is concerned with the interaction between an underexpanded supersonic gas jet and a flat plate with the

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